Generalized Catalan numbers and Gröbner bases

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- Fatemeh Mohammadi (KU Leuven).

Catalan numbers: Interpretations

Catalan numbers appear in a wide range of combinatorial and algebraic constructions, cf. e.g. [Bernhart 1999], [Aigner 1998].

- Parenthesis expressions
- Rooted trees
- Dyck paths
- Polygon dissections
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$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$$

Catalan numbers: Generalizations

Catalan numbers have been generalized in different directions, cf. e.g. [Stump et al. 2018], [Bergeron et al. 1996], [Linz 2022].

- Fuß–Catalan numbers
- W-Catalan numbers from Coxeter groups
- s-Catalan numbers via generalized binomial coefficients
- spin s-Catalan numbers
- q-Catalan numbers for weighted combinatorial objects

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We give lattice path interpretations of s-Catalan numbers.

s-binomial coefficients and s-Catalan numbers

The *s-binomial coefficients* $\binom{n}{d}_s$ are defined for $n, s \in \mathbb{N}$ via

$$(1 + x + \cdots + x^s)^n = \sum_{d=0}^{sn} \binom{n}{d}_s x^d \,.$$

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Definition (e.g. [Ghemit and Ahmia 2024, Linz 2022])

The *s-Catalan numbers* are given by

$$C_0^{(s)} = 1 \quad \text{and} \quad C_n^{(s)} = \binom{2n}{sn}_s - \binom{2n}{sn+1}_s \text{ for } n > 0.$$

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Our approach

We obtain lattice path characterizations of s-Catalan numbers, generalizing the well-known Dyck path description of Catalan numbers.

Algebra tools: Hilbert series and initial ideals

Work in $R = \mathbf{k}[x_1, \dots, x_n]$, standard graded polynomial ring over field of characteristic zero.

Definition

Let $M = \oplus_{d \geq 0} M_d$ be a finitely generated graded R-module with degree-d components M_d. The *Hilbert series* of M is

$$\mathrm{HS}(M; t) = \sum_{d=0}^{\infty} \dim_{\mathbf{k}}(M_d)\, t^d \, .$$

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Example

For $A = \mathbf{k}[x_1, x_2]/(x_1^2, x_2^3)$, we have $\mathrm{HS}(A; t) = 1 + 2t + 2t^2 + t^3$.

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Remark

Let $\{0\} \neq I \subseteq R$ be a homogeneous ideal, and $\prec$ a monomial ordering compatible with degrees. Then, $\mathrm{HS}(R/I; t) = \mathrm{HS}(R/\operatorname{in}_{\prec}(I); t)$.

Algebra tools: Weak and Strong Lefschetz properties

- We consider Artinian algebras $A = R/I$ with $I \subset R$ homogeneous and 0-dimensional, so $\mathrm{HS}(A;t)$ is a polynomial.

Definition

Let $\ell \in R$ be a general linear form. A has the WLP if, for all $d \in \mathbb{N}_0$,

$$\mu_{\ell,d} \colon A_d \to A_{d+1}, \quad f \mapsto \ell \cdot f$$

are maximal rank maps (i.e., are either injective or surjective).
Similarly, A has the SLP if, for all $d, e \in \mathbb{N}_0$,

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- It suffices to exhibit *one* linear form ℓ with these properties.
- If A is defined by a monomial ideal, then it suffices to consider $\ell = x_1 + \cdots + x_n$ [Nicklasson 2018].

Algebra tools: Almost complete intersections

Monomial ideals $P_{n,s+1} = (x_1^{s+1}, \ldots, x_n^{s+1}) \in \mathbb{Z}^n_{\geq 2}$ define Artinian complete intersection algebras $A_{n,s+1} = R/P_{n,s+1}$ that have the SLP.

Definition

Write $I_{n,s+1,k} = (x_1^{s+1}, \ldots, x_n^{s+1}, (x_1 + \cdots + x_n)^k) \subset \mathbf{k}[x_1, \ldots, x_n]$, where $k \geq 1$.

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- The algebra $A_{n,s+1} := \mathbf{k}[x_1, \dots, x_n]/P_{n,s+1}$ has s-binomial coefficients as the coefficients of its Hilbert series.
- The algebra $A_{n,s+1,k} := \mathbf{k}[x_1, \dots, x_n]/I_{n,s+1,k}$ has differences of s-binomial coefficients as the coefficients of its Hilbert series.

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Observation

An initial ideal of $I_{n,s+1,k}$ has the right Hilbert series, but how to obtain a description with lattice paths without computing the Gröbner basis?

Łukasiewicz paths

Write $\mathbf{m} = (s+1, \ldots, s+1) \in \mathbb{Z}^n_{\geq 2}$.

Definition

A monomial $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in R$ is called $\mathbf{m}$-*free* if $\alpha_i < m_i$ for $1 \leq i \leq n$.

From an $\mathbf{m}$-free monomial, we obtain an $\mathbf{m}$-admissible lattice path

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- $x_i^e \rightsquigarrow (i-1, b) \to (i, b-e+1) \quad$ ($(e-1)$-down).

Reflection and critical paths

One can introduce ([JLMO 2025]) a symmetry point on each vertical line $x = a$ with $0 \leq a \leq n$, depending on $\mathbf{m}$ and k. The construction also works for non-constant $\mathbf{m}$.

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$$^{i}\mathbf{P}' := (P_0, P_1, \ldots, P_{i-1}, P_i', P_{i+1}', \ldots, P_n')$$

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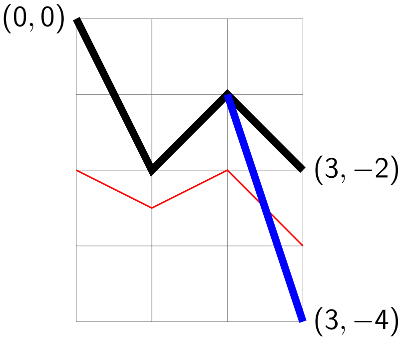
$$ \Lambda : \{\mathbf{P} \mid \mathbf{P} \text{ critical and } \mathbf{m}\text{-admissible}\} \to \{\mathbf{P} \mid \mathbf{P}\ \mathbf{m}\text{-admissible}\}, $$
$$ \mathbf{P} \mapsto {}^{\lambda(\mathbf{P})}\mathbf{P}'. $$

Critical path ideal

Let the critical path ideal be the monomial ideal generated by the monomials of the critical paths and the pure powers $x_i^{m_i}$.

Example

Consider the lattice path corresponding to the $(4, 2, 5)$-free monomial $x_1^3 x_3^2$, shown in black. Its image under the reflection map Λ is such that the initial two segments remain in black and the reflected portion appears in blue.



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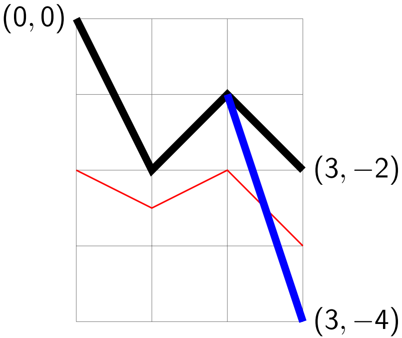
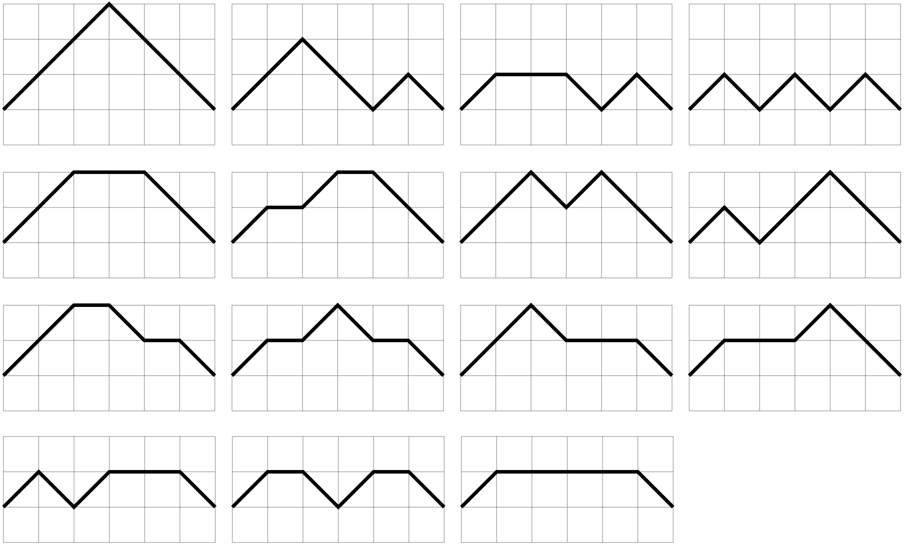


The critical path ideal has the correct Hilbert series, essentially because the reflection of critical paths of a degree d gives a surjection onto the set of critical paths of another degree d'. If $d < d'$, then all paths of degree d' are critical.

Example

We illustrate the lattice paths for $C_3^{(2)} = 15$.



s-Catalan triangle

Recall the *s-binomial coefficients* $\binom{n}{d}_s$, which satisfy

$$(1 + x + \cdots + x^s)^n = \sum_{d=0}^{sn} \binom{n}{d}_s x^d \,.$$

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The coefficients of the *s-Catalan triangle* are given by

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for $n > 0$ and $k \geq 0$. The s-Catalan triangle, written with entry $C_{n,k}^{(s)}$ in row n and column k, contains the s-Catalan numbers in its first column.

2-Catalan triangle

We give the first few rows of the 2-Catalan triangle. The 2-Catalan numbers are in its first column.

	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1	1								
2	3	6	6	3	1						
3	15	36	40	29	15	5	1				
4	91	232	280	238	154	76	28	7	1		
5	603	1585	2025	1890	1398	837	405	155	45	9	1

Figure: The 2-Catalan triangle.

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s-Catalan numbers

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2	2	3	4	5	6	7	8
3	5	15	34	65	111	175	260
4	14	91	364	1085	2666	5719	11096
5	42	603	4269	19845	70146	204687	518498
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Figure: The s-Catalan numbers for $n \leq 6$ and $s \leq 7$.

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Log-concavity and log-convexity

- A sequence $(a_n)_{n\geq 0}$ of nonnegative real numbers is (e.g. [Zhu 2013])
  - *log-concave* if $a_n^2 \geq a_{n-1}a_{n+1}$ for all $n \geq 1$
  - *log-convex* if $a_n^2 \leq a_{n-1}a_{n+1}$ for all $n \geq 1$
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- Open question in [Ghemit and Ahmia 2024]: Analyze log-convexity/concavity of rows/columns of any s-Catalan triangle.

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Other known results (see [Liu and Wang 2017] for more):

- The sequence of 1-Catalan numbers is log-convex.
- The sequence of Motzkin numbers is log-convex. [Aigner 1998; algebraic proof]; [Callan; combinatorial proof]
- The sequence of 2-Catalan numbers is log-convex [Ghemit and Ahmia 2024].

Log-concavity of the rows

Using our results, we provide the following lattice path interpretation of the numbers in the s-Catalan triangle for any positive integer s.

Proposition ([JLMO 2025])

For $m \geq 2$, $n \geq 1$, and $k \geq 0$, $C^s_{n,k}$ counts the $\mathbf{m}$-admissible lattice paths of degree $ns - k$ whose $\mathbf{m}$-free monomials are not in $\mathrm{in}(I_{2n,\mathbf{m},1})$.

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Corollary ([JLMO 2025])

The rows of the s-Catalan triangle are log-concave.

Injective proofs and encounter technique

We obtain finite sets whose cardinalities are the s-Catalan numbers.

Definition

For $s \geq 1$ and $n \geq 0$, define

$$\mathcal{C}_n^{(s)} = \{\mathbf{m}\text{-admissible non-critical paths ending at } (2n, n \cdot (2-s))\}\,.$$

We have $|\mathcal{C}_n^{(s)}| = C_{n,0}^{(s)}$.

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We have $|\mathcal{C}_n^{(s)}| = C_{n,0}^{(s)}$. Thus for showing log-convexity of the s-Catalan numbers, it is enough to construct an injective map

$$f_n : \mathcal{C}_n^{(s)} \times \mathcal{C}_n^{(s)} \to \mathcal{C}_{n-1}^{(s)} \times \mathcal{C}_{n+1}^{(s)}$$

for every integer $n \geq 1$.

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The *encounter technique* for Łukasiewicz paths in the sense of [Chen et al. 2021] constructs maps between sets of configurations of two paths by letting the first path follow the route of the second from the *first* encounter onwards and vice versa.

Encounter technique for 1-Catalan paths

Using encounters, one obtains injective maps

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thus giving another proof of the log-convexity of (1-)Catalan numbers.

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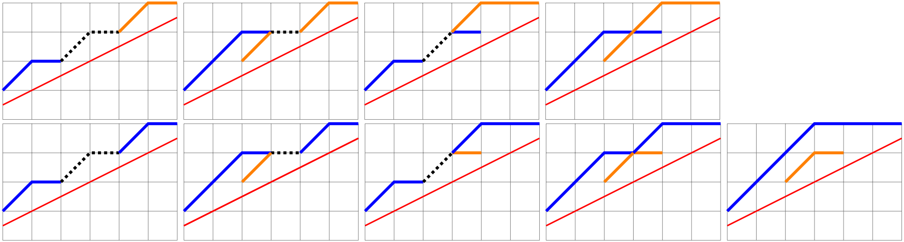
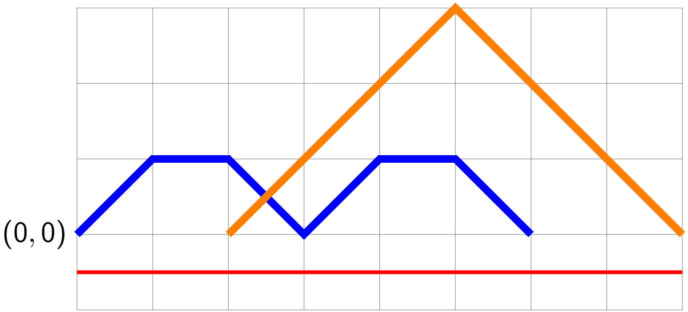


Admissibility restrictions

Here is an example where an injective proof using encounters fails. It is minimal in the sense that it is for paths in $\mathcal{C}_3^{(2)}$ and there is no counterexample for paths in $\mathcal{C}_n^{(1)}$ for any $n \geq 1$ nor for paths in $\mathcal{C}_n^{(2)}$ for $n \in \{1, 2\}$.



Open problem

The problem of deciding the log-convexity of s-Catalan numbers can be formulated as follows.

Open problem

Does there exist an injective map

$$f_n : \mathcal{C}_n^{(s)} \times \mathcal{C}_n^{(s)} \to \mathcal{C}_{n-1}^{(s)} \times \mathcal{C}_{n+1}^{(s)}$$

for every integer $n \geq 1$? If yes, give an explicit construction.

Outlook

- Our critical path ideals can be used to describe
  - Catalan numbers and their convolutions
  - Motzkin and Riordan numbers and their convolutions
  - Entries in s-Catalan triangles
  - ...and many other, also previously undescribed, sequences
- Relationships to
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Thank you